CS 331, Fall 2026

Mini-lecture 6 (10/23)

Today: Linear algebra

Notation

Vectors = lowercase letters

"column matrices"

$$v = \begin{pmatrix} 4 \\ 2 \\ -6 \end{pmatrix}$$

Matrices = uppercase letters

$$M = \begin{pmatrix} 2 & -7 \\ 3 & 1 \end{pmatrix}$$

Transpose = flip across diagonal (dimensions flip)

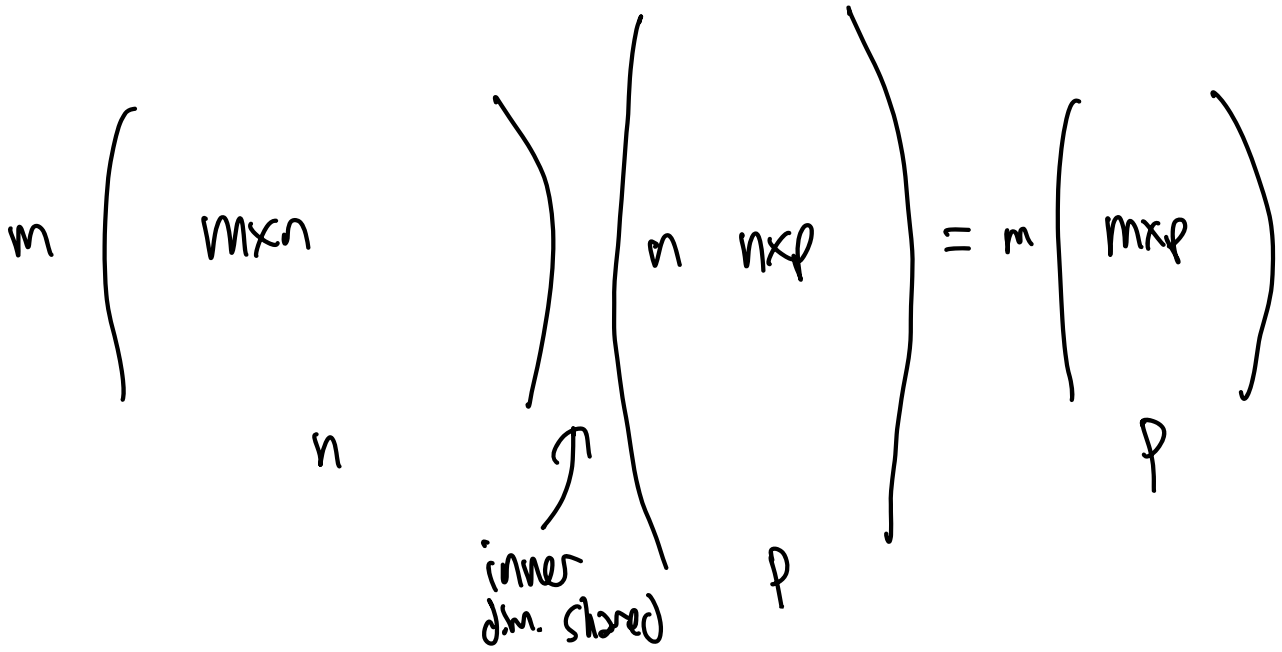
$$v^T = \begin{pmatrix} 4 & 2 & -6 \end{pmatrix}$$

$$M^T = \begin{pmatrix} 2 & 3 \\ -7 & 1 \end{pmatrix}$$

Symmetric:

$$M = M^T$$

Matrix multiplication recap



Easy case: $n=1$ $\quad u^T v \quad u, v \in \mathbb{R}^d$

$$\begin{pmatrix} 4 & -3 & 0 \end{pmatrix} \begin{pmatrix} 2 \\ -2 \\ 7 \end{pmatrix} = 4 \cdot 2 + -3 \cdot -2 + 0 \cdot 7$$

$$= 14$$

"dot product"
"inner product"

Bigger matrices: dot every row/col!

$$\begin{pmatrix} 7 & 6 & 6 \\ 4 & -3 & 0 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 12 & -2 \\ -2 & 1 & 4 \\ 7 & 3 & 6 \end{pmatrix} = \begin{pmatrix} 14 & & \\ & & \\ & & \end{pmatrix}$$

Special matrices/vectors:

e_i = i^{th} "standard basis" vector

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix} \qquad e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \end{pmatrix} \qquad e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ \vdots \end{pmatrix}$$

$$0_d = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \} \times d \qquad 1_d = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \} \times d$$

$$I_d = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$$

"identity matrix" in $\mathbb{R}^d$

Reason: $I_d v = v$ for all $v \in \mathbb{R}^d$ Ex: why?

Exercises: What is $e_i^T e_j$?

What is $\sum_{i \in [d]} e_i e_i^T$?

What is $u^T M v$?

Special name for $u^T u = \sum_{i \in [n]} u_i^2 = \|u\|_2^2$

We interpret $\| \ \|_2$ as length — Pythagorean theorem

Subspaces

Ex. Suppose $\begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ \\ \\ \end{pmatrix} = 7$

$\begin{pmatrix} 6 & 2 & 4 \end{pmatrix} \begin{pmatrix} x \\ \\ \\ \end{pmatrix} = -12$

What is ...

$\begin{pmatrix} 9 & 2 & 4 \end{pmatrix} \begin{pmatrix} x \\ \\ \\ \end{pmatrix} = ?$

$\begin{pmatrix} 6 & 14 & 28 \end{pmatrix} \begin{pmatrix} x \\ \\ \\ \end{pmatrix} = ?$

How can you tell? Redundant hints

Say I give you n equations in unknown $x \in \mathbb{R}^d$

$$a_1^T x = b_1$$
$$a_2^T x = b_2$$
$$\vdots$$
$$a_n^T x = b_n$$

"linear system"

What else do you know?

$a^T x$ for any

$$a = c_1 a_1 + c_2 a_2 + \dots + c_n a_n$$

We call all such a the Span of $a_1, \dots, a_n$

We say vectors $a_1\ a_2 \dots a_n$ are linearly independent

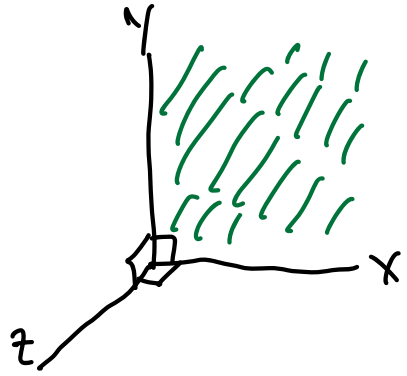
if $c_1 a_1 + c_2 a_2 + \dots + c_n a_n \neq 0_d \quad \forall \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

equiv. $a_i \notin \text{Span}\left(a_1\ a_2 \dots a_{i-1}\ a_{i+1} \dots a_n\right)$

Ex. What is $\text{Span}\left(\begin{pmatrix}1\\2\\0\end{pmatrix}, \begin{pmatrix}-2\\-3\\0\end{pmatrix}\right)$?

Are $\begin{pmatrix}2\\6\\3\end{pmatrix}$ $\begin{pmatrix}-7\\0\\5\end{pmatrix}$ $\begin{pmatrix}2\\0\\-2\end{pmatrix}$ linearly independent?

We say $S \subseteq \mathbb{R}^d$ is a <u>subspace</u> if it's the span of some set of vectors.



Observation: If $u \in S$, $v \in S$ then $\alpha u + \beta v \in S$ for any scalars α, β.

Proof: Write $S = \text{span}(v_1\ v_2\ \dots\ v_n)$

$u = c_1 v_1 + c_2 v_2 + \dots + c_n v_n$

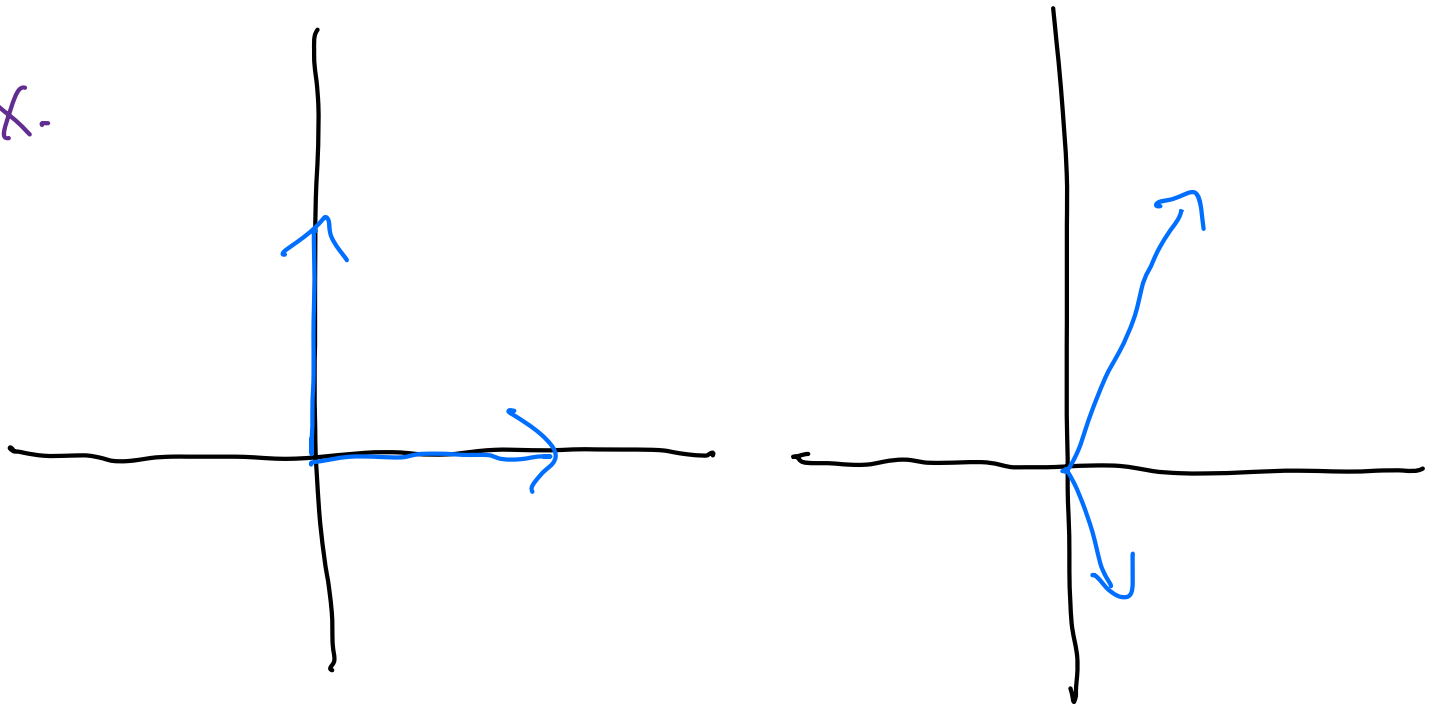
$v = c_1' v_1 + c_2' v_2 + \dots + c_n' v_n$

Then clearly $\alpha u + \beta v$ is also lincomb of $\uparrow$.

Fact: If $S \subseteq \mathbb{R}^d$ is a subspace,

- S is the span of any maximal linearly independent set of $\{v_1, v_2, \ldots, v_k\} \subseteq S$
  (can't add another vector w/o creating dependence)
- All maximal independent sets have same size
  (dimension of S, write $\dim(S) = k$)

Ex.



Same span... also same size $k = 2$

We call maximal set of linear independent vectors

$\{v_1, v_2, \ldots v_k\} \subseteq S$ a <u>basis</u> for S

How to choose a "nice" basis?

Recall that every $v \in S$ can be written

$$v = c_1 v_1 + c_2 v_2 + \ldots + c_k v_k$$

Ex. Suppose $S = \mathbb{R}^2 \subset \mathbb{R}^3$

Which of the following bases is easiest to determine c_1, c_2 for $v = \begin{pmatrix} 3 \\ 7 \\ 0 \end{pmatrix} \in S$?

Option 1: $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

Option 2: $\begin{pmatrix} 3/5 \\ 4/5 \\ 0 \end{pmatrix} \begin{pmatrix} -4/5 \\ 3/5 \\ 0 \end{pmatrix}$

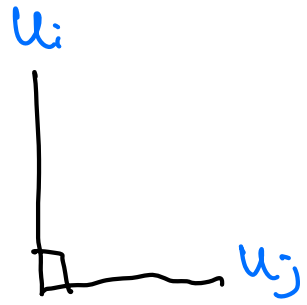
Option 3: $\begin{pmatrix} 2 \\ -2 \\ 0 \end{pmatrix} \begin{pmatrix} -2 \\ 3 \\ 0 \end{pmatrix}$

Orthonormal basis for $S \subseteq \mathbb{R}^d$, $\dim(S) = k$

- $\text{Span}(u_1\ u_2\ \dots\ u_k) = S$
- $\|u_i\|_2^2 = 1 \quad \forall i \in [k]$ "normal (-ized)"
- $u_i^T u_j = 0 \quad \forall i \neq j \in [k]$ "ortho (-gonal)"

recall it's $\|u_i\|_2 \|u_j\|_2 \cos\theta$

$\cos\theta = 0 \iff \theta = \pm\frac{\pi}{2}$



Concise formula: $U = (u_1\ u_2\ \dots\ u_k)$

$$U^T U = \begin{pmatrix} \vdots & & \vdots \\ \dots & u_i^T u_j & \dots \\ \vdots & & \vdots \end{pmatrix} = I_k$$

Observation: $UU^T v = v \quad \forall v \in S$

$$= \sum_{i \in [k]} (u_i^T v) u_i$$ easy coefficients